7/04/2021 Exam: 5/05/2021 or 12/05/2021

Part III, chap 2.

Zhu's algebra

V $\mathbb{Z}$-graded

Def: a, b homog. $a \in V_{\Delta_a}$, $b \in V_{\Delta_b}$

$$a \circ b = \mathrm{Res}_{z=0}\left(Y(a,z)\, b\, \frac{(z+1)^{\Delta_a}}{z^2} \right) = \sum_{i \geq 0} \binom{\Delta_a}{i} a_{(i-2)} b$$

extend $\circ$ by linearity.

$\mathrm{Zhu}(V) = V / V\circ V$, $V \circ V = \mathrm{span}_{\mathbb{C}}\{ a \circ b,\ a, b \in V\}$

thm: $\mathrm{Zhu}(V)$ is an associative alg, with unit $[|0\rangle]$ = image of $|0\rangle$ in $\mathrm{Zhu}(V)$

with product

$$a * b = \mathrm{Res}_z \left(Y(a,z)\, b\, \frac{(z+1)^{\Delta_a}}{z} \right) = \sum_{i \geq 0} \binom{\Delta_a}{i} a_{(i-1)} b$$

for homog a, b and extend by linearity.

proof: after several technical lemmas.

Lemma 1: $\forall\, a \in V$ homog, $\forall\, n \in \mathbb{Z}_{\geq 0}$, we have

$$T^n a = n! \binom{-\Delta_a}{n} a \quad \text{mod } V \circ V$$

proof: $a \circ |0\rangle = a_{(-2)}|0\rangle + \Delta_a a = Ta + \Delta_a a \Rightarrow Ta = -\Delta_a a$ mod $V \circ V$

this proves the statement for $n = 1$.

By induction, $\Delta_{Ta} = \Delta_a + 1$

$$T^{n+1} a = T^n (Ta) = n! \underbrace{\binom{-\Delta_a - 1}{i}} Ta \overset{(n=1)}{=} -\Delta_a a \quad \text{mod } V \circ V \quad \checkmark. \qquad \square$$

lemma 2 : $a, b \in V$ homog.

(1) $b * a = \mathrm{Res}_{z=0}\left(Y(a,z)\frac{(z+1)^{\Delta_a-1}}{z}\right) = \sum_{i\geq 0}\binom{\Delta_a-1}{i} a_{(i-1)}b$ mod $V\circ V$

mod $V\circ V$

(2) $a * b - b * a = \sum_{i\geq 0}\binom{\Delta_a-1}{i} a_{(i)}b$ mod $V\circ V$

In particular: the image of the conformal vector ω "belongs to the center of Zhu(V)"

proof : (1) $\Rightarrow$ (2)

$$a*b - b*a = \sum_{i\geq 0}\binom{\Delta_a}{i} a_{(i-1)}b - \sum_{i\geq 0}\binom{\Delta_a-1}{i} a_{(i-1)}b \quad \text{mod } V\circ V \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \sum_{i\geq 1}\binom{\Delta_a-1}{i-1} a_{(i-1)}b + a_{(-1)}b - a_{(-1)}b \quad \text{mod } V\circ V$$

$$= \sum_{i\geq 0}\binom{\Delta_a-1}{i} a_{(i)}b \quad \text{mod } V\circ V$$

$$\omega * a - a * \omega = \sum_{i\geq 0}\binom{1}{i}\omega_{(i)}a = Ta + \Delta_a a = a\circ|0\rangle$$

$$\omega_{(0)} = T, \quad \omega_{(1)} = H$$

(1)

fact : (skew-symmetry)

$$Y(a,z)b = e^{zT}Y(b,-z)a$$

proof (after).

(1) : $Y(b,z)a = e^{zT}Y(a,-z)b = \sum_{n\in\mathbb{Z}} e^{zT}a_{(n)}b(-z)^{-n-1}$

lemma 1 $\to$ $= \sum_n\sum_{j\geq 0}\frac{T^j}{j!}(a_{(n)}b)\, z^j(-z)^{-n-1}$ mod $V\circ V$

$= \sum_n\sum_j\binom{-\Delta_a-\Delta_b+n+1}{j} z^j a_{(n)}b(-z)^{-n-1}$ "

$= \sum_n(-z)^{-n-1}(z+1)^{-\Delta_a-\Delta_b+n+1}a_{(n)}b$ "

$= (z+1)^{-\Delta_a-\Delta_b}\, Y\left(a, -\frac{z}{z+1}\right)b.$ "

therefore : $b * a = \mathrm{Res}_{z=0}\left(Y(b,z)\frac{(z+1)^{\Delta_b}}{z}\right) = \mathrm{Res}_z\, Y\left(a, -\overset{f(z)}{\overset{\shortparallel}{\frac{z}{z+1}}}\right)b\,\frac{(z+1)^{\cancel{\Delta_b}}}{z}(z+1)^{-\Delta_a-\cancel{\Delta_b}}$

Recall: $g(w) = \sum_m v_m w^m \in V((w))$, $f(z) = \sum a_n z^n \in z\mathbb{C}[[z]]$, $a_1 \neq 0$

$$\mathrm{Res}_w\, g(w) = \left(\mathrm{Res}_z\, g(f(z))\right) \frac{\partial}{\partial z} f(z) \qquad (*)$$

$$g(f(z)) = \sum_{n \geq M} v_m (f(z))^m = \sum_{n \geq M} \sum_{j \geq 1} v_m (a_1 z)^n \binom{m}{j} \bar{f}^j$$

$$f(z) = a_1 z + \sum_{i \geq 2} a_i z^i = a_1 z\Big(1 + \underbrace{\sum_{i=2}^{\infty} \frac{a_i}{a_1} z^{i-1}}_{\bar{f}}\Big)$$

Using $(*)$ with $w = -\frac{z}{z+1}$

$$b * a = \mathrm{Res}_w\, Y(a,w) \frac{(w+1)^{\Delta_a - 1}}{w} \quad \text{mod } O(V).$$

[$w = \frac{-z}{z+1} \iff z = -\frac{w}{w+1}$ . . .]

It remains to show the skew-symm. formula.

$$Y(a,z)b = e^{zT} Y(b,-z)a$$

Recall: $Y(b,w)|0\rangle = e^{wT} b = \sum_{n \geq 0} \frac{T^n b}{n!} w^n \qquad (\cdot)$

By locality + $(\cdot)$, for $N \gg 0$

$$(z-w)^N Y(a,z) e^{wT} b = (z-w)^N Y(a,z) Y(b,w)|0\rangle = (z-w)^N Y(b,w) Y(a,z)|0\rangle$$
$$= (z-w)^N Y(b,w) e^{zT} a$$

Next:

$$e^{zT} Y(b,w) e^{-zT} = \overbrace{\sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}(zT)^n}^{\exp(\mathrm{ad}(zT))\, Y(b,w)} Y(b,w)$$

[general fact: $e^x y e^{-x} = e^{(\mathrm{ad}\, x)} y$, $x, y \in \mathrm{End}(V)$

where: $(\mathrm{ad}\, x)(y) = [x,y] = xy - yx$

write: $\mathrm{ad}\, x = \lambda_x + \rho_{-x}$, λ_x: left mult., ρ_x right mult.

λ_u, ρ_{-u} commute each others

So: $\exp(\mathrm{ad}\, u) = \exp(\lambda_u)\exp(\rho_{-u}) \quad \ldots\,)$

$$(z-w)^N Y(a,z) e^{wT} b = (z-w)^N Y(b,w) e^{zT} a$$

Therefore:

generalized Taylor formula.

$$e^{zT} Y(b,w) e^{-zT} = \sum_{n\geq 0} \frac{1}{n!} (\mathrm{ad}\, zT)^n (Y(b,w)) \underset{\nearrow}{=} \sum_{n\geq 0} \frac{1}{n!} z^n \partial_w^n Y(b,w) \overset{!}{=} Y(b, \underbrace{z+w})$$

$[T, Y(b,w)] = \partial_w Y(b,w)$

$(z+w)^{-1}$ means $\tau_{z,w}\left(\frac{1}{z+w}\right)$

Hence: $(z-w)^N Y(a,z) e^{wT} b = (z-w)^N \underset{\overset{\shortparallel}{Y(b,(w-z)+z)}}{Y(b,w)} e^{zT} a = (z-w)^N e^{zT} Y(b, w-z) a$

Since on the left-hand side there is no negative power in w, one can set $w=0$

$\Longrightarrow Y(a,z) b = e^{zT} Y(b,-z) a.$ $\quad \square$

Lemma 3: $a \in V$ homog, $m \geq n \geq 0$

$$\mathrm{Res}_z\, Y(a,z) \frac{(z+1)^{\Delta_a+n}}{z^{2+m}} b \in V \circ V$$

Pf: Since: $\displaystyle \frac{(z+1)^{\Delta_a+n}}{z^{2+m}} = \sum_{i=0}^{n} \binom{n}{i} \frac{(z+1)^{\Delta_a}}{z^{2+m-i}}$ $\quad m-i \geq 0$ since $m \geq n$.

So it suffices to prove the lemma for $n=0$ and $m \geq 0$.

Induction on m.

* $m=0$ clear by definition of $V \circ V$.

* assume the lemma true for $m \leq k$. Prove it for $m=k+1$

By induction $\displaystyle \mathrm{Res}_z\,(T_a, z) = \frac{(z+1)^{\Delta_a+1}}{z^{2+k}} b \in V \circ V$ $\qquad \Delta_{Ta} = \Delta_a + 1$

$$\mathrm{Res}_z\,(T_a z) \frac{(z+1)^{\Delta_a+1}}{z^{2+k}} b = \mathrm{Res}_z\left(\frac{\partial}{\partial z} Y(a,z) \frac{(z+1)^{\Delta_a+1}}{z^{2+k}} b\right)$$

$$= -\,\mathrm{Res}_z\left(Y(a,z) \frac{\partial}{\partial z}(\qquad) b\right)$$

$$= -(\Delta_a+1)\underbrace{\mathrm{Res}_z\left(Y(a,z) \frac{(z+1)^{\Delta_a}}{z^{2+k}} b\right)}_{\in V \circ V \text{ by induction}} + (2+k)\,\mathrm{Res}_z\, Y(a,z) \frac{(z+1)^{\Delta_a}}{z^{2+k+1}} b$$

$\Rightarrow$ statement for $m = k+1$ $\qquad$ □

proof of the theorem.

first: for homog a, $a * |0\rangle = a_{(-1)}|0\rangle = a$, $|0\rangle * a = |0\rangle_{(-1)} a = a$

To prove the theorem, we have to show that $V\circ V$ is a two-sided ideal of $Zhu(V) = V/_{V\circ V}$ so that $*$ is well-defined on $Zhu(V)$ and that $*$ is associative:

It suffices to show

(1) $a*(V\circ V) \subset V\circ V$

(2) $(V\circ V) * a \subset V \circ V$

(3) $(a*b)*c - a*(b*c) \in V\circ V$

We detail only (1), others are similar:

(1) a, b, c homogeneous in V

Let us show that $a*(b\circ c) - \underbrace{b\circ(a*c)}_{\in V\circ V} \in V\circ V \implies a*(b\circ c) \in V\circ V$

Rem: Borcherds identities (1) & (2) ($[a_{(m)}, b_{(n)}] = \dots$, $(a_{(m)}b)_{(n)} = \dots$)

$\iff$

$$\sum_{i\geq 0} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)} = \sum_{i\geq 0} (-1)^i \binom{r}{i} a_{(p+r-i)} b_{(q+i)} - (-1)^r b_{(q+r-i)} a_{(p+i)}\Big) \qquad (3)$$

(we recover (1) & (2) with $r=0$, $p=0$)

$\iff$

$$\operatorname{Res}_{z-w} Y(Y(a,z-w)b,w)\, \iota_{w,z-w} F(z,w)$$

$F(z,w) = z^p w^q (z-w)^r$

$$= \operatorname{Res}_z Y(a,z) Y(b,w)\, \iota_{z,w} F(z,w) - \operatorname{Res}_z Y(b,w) Y(a,z)\, \iota_{w,z} F(z,w)$$

From B'n (3), we get:

$$a * (b \circ c) - b \circ (a * c) = \mathrm{Res}_z\, Y(a,z)\frac{(z+1)^{\Delta_a}}{z}\, \mathrm{Res}_w\, Y(b,w)\frac{(w+1)^{\Delta_b}}{w^2}\, c$$

$$- \mathrm{Res}_w\, Y(b,w)\frac{(w+1)^{\Delta_b}}{w^2}\, Y(a,z)\frac{(z+1)^{\Delta_a}}{z}\, c$$

$$= \mathrm{Res}_w\, \mathrm{Res}_{z-w}\, Y(Y(a,z-w)b,w)\frac{(z+1)^{\Delta_a}}{z}\,\frac{(w+1)^{\Delta_b}}{w^2}\, c$$

$$= \underbrace{\sum_{i=0}^{\Delta_a}\sum_{j\geq 0}\binom{\Delta_a}{i}\, \mathrm{Res}_w\, Y(a_{(i+j)}b,w)(-1)^j\,\frac{(w+1)^{\Delta_a+\Delta_b-1}}{w^{j+2}}\, c}_{\in V\circ V \text{ by lemma 3.}}$$

$\Delta_{a_{(i+j)}b} = \Delta_a+\Delta_b-i-j-1$

□.

Modules over V

A V-module M is called a positive energy representation if $M = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} M_{\lambda+n}$,

where $M_\lambda \neq 0$, $M_d = \{m\in M : Hm = dm\}$

grading comes from Hamiltonian H. (ex: $H = L_0$ if V conformal).

$M_{top} := M_\lambda$ top degree component

$$\boxed{a_{(n)} M_d \subset M_{d+\Delta_a-n-1}}$$

$$\overset{\shortparallel}{a^M_{(n)}}\, M_d$$

$$[H, a^M_{(n)}] = -(n+1)\,a^M_{(n)} + (Ha)^M_{(n)}$$

$$\overset{\shortparallel}{H a^M_{(n)} - a^M_{(n)} H}$$

$a\in V$ homog. $o(a) := a_{(\Delta_a-1)} = a^M_{(\Delta_a-1)}$

Then $o(a)$ preserves homog. components of M

$$a_{(\Delta_a-1)} M_d \subset M_d.$$

Thm (Zhu)

For any positive energy rep M of V, $\overset{\substack{\text{image of } a\in V \text{ in } \mathrm{Zhu}(V)\\ \downarrow}}{[a]} \longmapsto o(a)$ gives a well-defined representation of $\mathrm{Zhu}(V)$ on M_{top}.

Moreover, the correspondence $M \longmapsto M_{\mathrm{top}}$ gives a bijection between the set of isom classes of irreducible positive energy reps of V and that of simple $\mathrm{Zhu}(V)$-modules.

Pf: later.

§2 - Relations with $\overset{\mathrm{Zhu}(V)}{\text{Zhu's algebra}}$ and Zhu's $\overset{R_V}{C_2\text{-alg.}}$

$V = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} V_\Delta$, $\quad V_{\leq p} := \bigoplus_{\Delta=0}^{p} V_\Delta \quad \rightsquigarrow$ increasing filtration on V.

Set $\mathrm{Zhu}_p(V) := \mathrm{Im}(V_{\leq p} \longrightarrow \mathrm{Zhu}(V))$

$0 = \mathrm{Zhu}_{-1}(V) \subset \mathrm{Zhu}_0 V \subset \dots \subset \mathrm{Zhu}_p V \subset \dots$, $\quad \mathrm{Zhu}(V) = \bigcup_{p \geq 0} \mathrm{Zhu}_p(V)$

$$\mathrm{gr}\,\mathrm{Zhu}\,V = \bigoplus_{p \geq 0} \mathrm{Zhu}_p V / \mathrm{Zhu}_{p-1} V$$

. Since $a_{(n)} b \in V_{\Delta_a + \Delta_b - n - 1}$ homog a, b

$\Rightarrow \mathrm{Zhu}_p(V) * \mathrm{Zhu}_q(V) \subset \mathrm{Zhu}_{p+q} V \qquad \rightsquigarrow$ gr Zhu V graded alg.

$$a * b = \sum_{i \geq 0} \binom{\Delta_a}{i} \underbrace{a_{(i-1)} b}_{\in V_{\Delta_a + \Delta_b - i}}$$

. $a * b - b * a = \sum_{i \geq 0} \binom{\Delta_a - 1}{i} a_{(i)} b = a_{(0)} b + (\Delta_a - 1) a_{(1)} b + \dots$

$[\mathrm{Zhu}_p V, \mathrm{Zhu}_q V] \subset \mathrm{Zhu}_{p+q\underline{\underline{-1}}} V$

So $\quad$ gr Zhu V is a commutative alg (even "Poisson alg")

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ \mathrm{gr}\,\mathrm{Zhu}\,V & & R_V \text{ comm alg (Poisson)} \end{tikzcd}$$

Prop there is a well-defined surjective (Poisson) alg homom:

$$\eta_V : R_V \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}\,V$$

$$\underset{a + C_2 V}{\overline{a}} \longmapsto a \bmod V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta \qquad \text{for homog } a$$

$$\Rightarrow (\mathrm{Spec}\,\mathrm{gr}\,\mathrm{Zhu}\,V)_{red} \subset (\mathrm{Spec}\,R_V)_{red} = X_V$$

Rem: η_V NOT an isomorphism in general

ex: $\mathfrak{g} = E_8$, $V = L_1(\mathfrak{g})$ then η_V NOT isom.

Conjecture: if V is a simple $\mathbb{Z}_{\geq 0}$ - graded conformal VA, then $X_V = (\mathrm{Spec}\ \mathrm{gr}\ \mathrm{Zhu})_{\mathrm{red}}$

$$\begin{tikzcd}
V \arrow[r] \arrow[dd] \arrow[ddr, dashed] & \mathrm{gr}^F V \simeq \mathrm{gr}_G V \arrow[d, "\mathrm{Zhu}\ C_2."] \\
 & R_V = V/C_2 V \arrow[d] \\
\mathrm{Zhu}\, V. \arrow[r, "\mathrm{gr}"'] & \mathrm{gr}\ \mathrm{Zhu}\, V
\end{tikzcd}$$

$$\underbrace{(\mathrm{Spec}\ \mathrm{gr}\ \mathrm{Zhu}\, V)_{\mathrm{red}} \subset X_V \underline{\underline{\subseteq}}}\ SS(V)_{\mathrm{red}}$$

Proof: If $\overset{a_{(-2)}b}{v} \in C_2 V$, then $v \in V \circ V + \bigoplus_{\Delta < \Delta_v} V_\Delta$

$C_2 V = \mathrm{Span}_{\mathbb{C}} \langle \underset{\deg\,=\,\Delta_a+\Delta_b+1}{a_{(-2)}b}\ ,\ a, b \text{ homog} \rangle$

$$\underset{\substack{\nearrow \\ V\circ V}}{a \circ b} = \sum_{i \geq 0} \binom{\Delta_a}{i} a_{(i-2)} b = \underline{\underline{a_{(-2)} b}} + \underbrace{\sum_{i\geq 1} \binom{\Delta_a}{i} \overbrace{a_{(i-1)} b}}_{\bigoplus_{\Delta < \Delta_b+\Delta_a+1} V_\Delta}$$

$$\deg (a_{(i-1)}b) = \underline{(\Delta_a+\Delta_b+1)} - i$$

$$\Rightarrow \overset{\overset{v}{\shortparallel}}{a_{(-2)} b} = a \circ b \mod \bigoplus_{\Delta < \Delta_v} V_\Delta \qquad \boxed{}$$

Examples:

$$J_\infty R_V \overset{\sim}{\twoheadrightarrow} \mathrm{gr}^F V$$

Recall: a VA <u>admits</u> a PBW basis if R_V is a polynomial ring and $J_\infty R_V \simeq \mathrm{gr}^F V$

(the original definition: V admits a PBW basis if $\exists \{a^i : i \in \underset{\text{ordered set}}{I}\}$ st

the set $a^{i_1}_{(n_1)} \cdots a^{i_r}_{(n_r)} |0\rangle$, and $|0\rangle$, $i_1 \leq i_2 \leq \cdots \leq i_r$

if $n_s \leq n_{s+1}$ if $i_s = i_{s+1}$

form a basis of V.

<u>Ex</u>: $V^k(\mathfrak{g})$

thm: if V admits a PBW basis, then $\eta_V : R_V \longrightarrow \operatorname{gr} \mathrm{Zhu}\, V$ is an isom.

Application: $V = V^k(\mathfrak{g})$ admits a PBW basis.

$$\eta_V : \underset{\overset{\shortparallel}{S(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]}}{R_V} \xrightarrow{\sim} \operatorname{gr} \mathrm{Zhu}\, V$$

$$U(\mathfrak{g}) \xrightarrow{\Phi} \mathrm{Zhu}\, V^k(\mathfrak{g})$$ well-defined. alg homom.

$$\mathfrak{g} \ni x \longmapsto [x_{(-1)}|0\rangle]$$

$$\underline{x * y - y * x} = \sum_{i \geq 0} \binom{\overset{\leq 0}{\Delta_x - 1}}{i} x_{(i)} y = x_{(0)} y = \underline{[x, y]}$$

$x, y \in \mathfrak{g} \subset V^k(\mathfrak{g})$, $x \longmapsto x_{(-1)}|0\rangle$

. this map respects the filtration on both sides: $\Phi(U_p \mathfrak{g}) \subset \mathrm{Zhu}_p V^k \mathfrak{g}$

$\rightsquigarrow$ $$\operatorname{gr} \Phi : \underset{\simeq S(\mathfrak{g})}{\operatorname{gr} U(\mathfrak{g})} \xrightarrow{\sim} \operatorname{gr} \mathrm{Zhu}\, V^k(\mathfrak{g}) \simeq R_V$$

$$x \longmapsto \overline{x_{(-1)}|0\rangle}$$

$\Rightarrow$ Φ is an isom.

cond: $\mathrm{Zhu}\, V^k(\mathfrak{g}) \simeq U(\mathfrak{g})$ $\leftarrow$ DOES NOT depend on k.

Zhu(?) is right-exact.

$N \to M \to 0$.

$\mathrm{Zhu}\, L_k(\mathfrak{g})$?? Difficult in general. , $L_k(\mathfrak{g}) = V^k(\mathfrak{g}) / N_k$

What we know: $\mathrm{Zhu}\, L_k(\mathfrak{g}) = U(\mathfrak{g}) / J_k$, where J_k is the image of N_k in $\underset{\simeq U(\mathfrak{g})}{\mathrm{Zhu}\, V^k(\mathfrak{g})}$

$\nwarrow$ here it depends on k.

Rem: When $\eta_{L_k(\mathfrak{g})}$ is an isom, $X_{L_k(\mathfrak{g})}$ is really an analogue to the associated variety of primitive ideals of $U(\mathfrak{g})$

$=$ zero locus of $\underline{\operatorname{gr} J_k}$.

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$\mathrm{Zhu}(V) = V / V\circ V$: assoc alg.

$\mathrm{Zhu}(V^k(g)) = U(g)$

$L_k(g) = V^k(g)/N_k$ simple VA

$\mathrm{Zhu}(L_k(g))$ is a quotient of $\mathrm{Zhu}(V^k(g)) = U(g)$

$0 \longrightarrow N \xrightarrow{\iota} V \xrightarrow{\pi} L \longrightarrow 0$

V : VA, $N \subset V$ ideal of V, $L = V/N$ VA from the VA structure on V

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & L \arrow[r] & L/L\circ L \\
V/V\circ V \arrow[urr, two heads, "??"'] & &
\end{tikzcd}$$

$\pi(a\circ b) = \sum_{i\geq 0} \binom{\Delta_a}{i} a_{(i-2)} b$ mod N

$= \sum_{i\geq 0} \binom{\Delta_a}{i} (a+N)_{(i-2)} (b+N)$ mod N $\quad N \subset V$ ideal of V

$\mathrm{Zhu}(L)$ is a quotient of $\mathrm{Zhu}(V)$

In fact the kernel of $\mathrm{Zhu}(V) \twoheadrightarrow \mathrm{Zhu}(L)$ is $N/V\circ N$: image in $\mathrm{Zhu}(V)$ of the image $N \longrightarrow V \longrightarrow \mathrm{Zhu}(V)$.

Rem: M a vertex V-module

$\mathrm{Zhu}(M) := M/V\circ M$, where $V\circ M = \mathrm{span}_{\mathbb{C}}\{ a\circ m := \sum_{i\geq 0} \binom{\Delta_a}{i} a^M_{(i-2)} m^{\subset M} \}$

Fact: $\mathrm{Zhu}(M)$ is a bimodule of $\mathrm{Zhu}(V)$ with:

$a * m = \underline{\sum_{i\geq 0} \binom{\Delta_a}{i} a^M_{(i-1)} m}$, $\quad m * a = \sum_{i\geq 0} \binom{\Delta_a - 1}{i} a^M_{(i-1)} m$

$b * a =$

We have a functor: $V\text{-Mod} \longrightarrow \mathrm{Zhu}(V)\text{-liMod}$, $M \longmapsto \mathrm{Zhu}(M)$

this functor is right-functor.

Rem: We have similar remarks for $R_V = V/C_2(V)$

If M is a V-module:

$C_2(M) = \mathrm{Span}_{\mathbb{C}}\{ a^{M}_{(-2)} m : a \in V, m \in V \} = F^1 M$

(we can define similarly $F^p M$.)

$\bar{M} := M/C_2(M)$

$\tilde{X}_M := \mathrm{supp}_{R_V}(\bar{M}) = \{ p \in \overset{\overset{\tilde{X}_V}{\|}}{\mathrm{Spec}\, R_V} : p \supset \mathrm{Ann}_{R_V}(\bar{M}) \} \subset \tilde{X}_V$

R_V acts on $\bar{M}$

Exercise: Compute $\mathrm{Zhu}(D^{ch}_{G,k})$

(1) $D^{ch}_{G,k}$ admits a PBW basis

(2) $R_{D^{ch}_{G,k}} = \mathbb{C}[T^*G]$

§3 – Current algebra and universal algebra of vertex algebras

Lemma: V/TV is a Lie algebra by

$$[a+TV,\, b+TV] = a_{(0)}b + TV \qquad a, b \in V$$

proof: (1) skew-symmetry < skew-symmetry for V : $Y(a,z)b = e^{zT}Y(b,-z)a$

$$\Longrightarrow a_{(0)}b = -b_{(0)}a \mod TV$$

(2) Jacobi identity: < Borcherds identity

$$[a+TV,\, [b+TV,\, c+TV]] = b_{(0)}(a_{(0)}c) + [a_{(0)}, b_{(0)}] \quad \text{mod } TV.$$

∎

Exercise: (related to this construction).

$R_V = V/C_2(V)$: comm alg with : $\bar{a}\cdot\bar{b} = \overline{a_{(-1)}b}$, where $\bar{a}$ = image of a in R_V

Set $\{\bar{a}, \bar{b}\} := \overline{a_{(0)}b} = a_{(0)}b \mod C_2(V)$

Show that $\{\cdot,\cdot\}$ is a Lie bracket and that it verifies the Leibniz rule:

$$\{\bar{a}\cdot\bar{b}, \bar{c}\} = \bar{a}\cdot\{\bar{b},\bar{c}\} + \{\bar{a},\bar{c}\}\cdot\bar{b}$$

We say that $(R_V, \{\cdot,\cdot\})$ is a Poisson algebra.

Lemma: (R,∂) diff alg (ie: a comm VA)

$$\mathrm{Lie}(V,R) := V\otimes R \big/ (T\otimes 1 + 1\otimes\partial)(V\otimes R) \quad \text{is a Lie alg by}$$

$$[a\otimes r,\, b\otimes s] = \sum_{j\geq 0} a_{(j)}b \otimes \left(\frac{1}{j!}\partial^j r\right) s$$

proof: R is a comm VA, $V\otimes R$ is a VA with translation operator $T\otimes 1 + 1\otimes\partial$.

$b\otimes s$

and $Y(a\otimes r, z) = Y(a,z)\otimes Y(r,z) = \sum_{n,m}(a_{(n)}\otimes r_{(m)})\,(b\otimes s)\, z^{-(n+m+1)-1}$

$[a\otimes r, b\otimes s] = (a\otimes r)_{(0)}(b\otimes s) + \tilde{T}\, V\otimes R$ $\qquad \tilde{T} = T\otimes 1 + 1\otimes\partial$

$$= \sum_n (a_{(n)}\otimes r_{(-n-1)})(b\otimes s) = \sum_n a_{(n)}b \otimes \underset{\overset{\shortparallel}{(\frac{1}{j!}\partial^j r)s.}}{r_{(-n-1)}s}$$

$r_{(n)}=0 \quad n\geq 0$ since R is comm. $\square$

Df: The (Borcherds) Lie algebra associated with a vertex algebra V is by definition the Lie algebra $\mathrm{Lie}(V) := \mathrm{Lie}(V, \mathbb{C}[t,t^{-1}]) = V\otimes\mathbb{C}[t,t^{-1}] \big/ (T\otimes 1 + 1\otimes\partial_t)\, V\otimes\mathbb{C}[t,t^{-1}]$

where $\mathbb{C}[t,t^{-1}]$ is viewed as a diff. alg with derivation ∂_t.

Set $a_{\{n\}}$ the image of $a\otimes t^n \in V\otimes\mathbb{C}[t,t^{-1}]$ in $\mathrm{Lie}(V)$

We have: $[a_{\{m\}}, a_{\{n\}}] = \sum_{j\geq 0}\binom{m}{j}(a_{(j)}b)_{\{m+n-j\}}$

Moreover by definition, $(Ta)_{\{n\}} = -n\, a_{\{n-1\}}$. $\quad Ta\otimes t^n = -n a_{(n-1)}\otimes t^n$

$$[a\otimes t^m, b\otimes t^n] = \sum_{j\geq 0} a_{(j)}b\otimes\left(\frac{1}{j!}\partial^j t^m\right)t^n = \sum_{j\geq 0}\binom{m}{j}(a_{(j)}b)\, t^{m+n-j}$$

$$\shortparallel\ \frac{m!}{(m-j)!\,j!}t^{m-j}t^n$$

$$\shortparallel\ \binom{m}{j}$$

Recall: M is a V-module iff $\exists\ V\longrightarrow\mathcal{F}(M)$, $a\longmapsto Y_M(a,z)$ linear map st:

(1) $Y_M(|0\rangle, z) = \mathrm{Id}_M$

(2) $[Y_M(a,z), Y_M(b,z)] = \sum_{j\geq 0} Y_M(a_{(j)}b, w)\,\frac{1}{j!}\partial_w^j\,\delta(z-w)$

(3) $Y_M(a_{(n)}b, z) = Y_M(a,z)_{(n)}Y(b,z)$ $\qquad$ for all $a, b\in V$.

Lemma: Any V-module M is a $\mathrm{Lie}(V)$-module by $a_{\{n\}} \longmapsto a^M_{(n)} = a_{(n)}$ for $a \in V$, $n \in \mathbb{Z}$.

$$[a_{\{m\}}, b_{\{n\}}] = \sum_j \binom{m}{j} (a_{(j)}b)_{\{m+n-j\}} \longmapsto \sum_j \binom{m}{j} (a_{(j)}b)^M_{(m+n-j)}$$
$$= [a^M_{(m)}, a^M_{(n)}].$$

⚠ A $\mathrm{Lie}(V)$-module no need to be a V-module since identities (2) and (3) may be NOT satisfied.

$$(a^M_{(n)} b)_{(n)} c \ldots$$

In order to make sense of $(a_{\{m\}} b)_{\{n\}} = \ldots$ we introduce a completion of $U(\mathrm{Lie}(V))$.

- Continue to assume that V is $\mathbb{Z}$-graded by H. Then $\mathrm{Lie}(V)$ is $\mathbb{Z}$-graded by $(\mathrm{ad} H)(a_{\{n\}}) = -(n+1)a_{\{n\}} + (Ha)_{\{n\}}$

We have: $\mathrm{Lie}(V) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Lie}(V)_d$, $\mathrm{Lie}(V)_d = \{x \in \mathrm{Lie}(V) : (\mathrm{ad} H)x = dx\}$

Let $U(\mathrm{Lie}(V)) = \bigoplus_d U(\mathrm{Lie}(V))_d$ be the induced $\mathbb{Z}$-grading

Define: $\widetilde{U(\mathrm{Lie} V)} = \bigoplus_{d \in \mathbb{Z}} \widetilde{U(\mathrm{Lie}(V))}_d$

where: $$\widetilde{U(\mathrm{Lie}(V))}_d = \varprojlim_{\substack{r \in \mathbb{Z} \\ r \to \infty}} U(\mathrm{Lie}(V))_d \Big/ \Big(\sum_{p \leq r} U(\mathrm{Lie} V)_{p-r}\, U(\mathrm{Lie}(V))_p\Big)$$

$\widetilde{U(\mathrm{Lie} V)}$ complete topological ring.

Now the identity

$$(a_{\{m\}} b)_{\{n\}} = \sum_{j \geq 0} (-1)^j \binom{m}{j} \left(a_{\{m-j\}} b_{\{n+j\}} - (-1)^r b_{\{m+n-j\}} a_{\{j\}}\right) \qquad (B2)$$

makes sense in $\widetilde{\mathcal{U}(\mathrm{Lie}\,V)}$

let $I = \bigoplus_d I_d$ be the graded ideal of $\widetilde{\mathcal{U}(\mathrm{Lie}\,V)}$ generated by (B2) and

$|0\rangle_{\{n\}} = \delta_{n,-1}$

Let $\mathcal{U}(V) = \bigoplus_{d\in\mathbb{Z}} \mathcal{U}(V)_d$, where $\mathcal{U}(V)_d = \widetilde{\mathcal{U}(\mathrm{Lie}\,V)}_d \big/ \overline{I_d}$ ← closure in $\widetilde{\mathcal{U}(\mathrm{Lie}\,V)}_d$

Def: $\mathcal{U}(V)$ is called the enveloping algebra of V, or current algebra of V.

| complete ring.

A $\mathcal{U}(V)$-module M is called smooth if the action $\mathcal{U}(V)\times M \to M$ is smooth where M is equipped with the discrete topology.

Hence: A V-module is the same as a smooth $\mathcal{U}(V)$-module

New interpretation of Zhu(V)

$U(V)_0$ is a subalg of $U(V)$

$a, b \in U(V)_0 \qquad (\mathrm{ad}\, H)(ab) = 0$

$U(V)_p \, U(V)_q \subset U(V)_{p+q}$.

Define : $A(V) = U(V)_0 \Big/ \overline{\sum_{r>0} U(V)_r \, U(V)_{-r}}$

thm 2 We have an isom of algebras : $\mathrm{Zhu}(V) \simeq A(V)$

proof : later.

proof of thm 1 : Let $U(V)_{\le 0} = \bigoplus_{d \le 0} U(V)_d \subset U(V)$.

For an $A(V)$-module E, define the positive energy representation of V

$$\mathrm{Ind}_{A(V)}^{U(V)} E = U(V) \otimes_{U(V)_{\le 0}} E$$

graded by $U(V)_d \otimes_{U(V)_{\le 0}} E$

$U(V)_{\le 0}$ acts on E by the projection

$$U(V)_{\le 0} \xrightarrow{\pi} A(V)$$

Let M be a simple positive energy representation of V, $M = \bigoplus_{n \in \mathbb{Z}_{\ge 0}} M_{\lambda+n}$, $\overset{M_{top}}{\overset{\shortparallel}{M_\lambda}} \neq 0$

then M_{top} is a simple $U(V)_0$-module on which $U(V)_{-r}$ acts trivially $r>0$

($a_{\{n\}} m \in M_{\Delta_a - n - 1 + \lambda}$ if $m \in M_{top} = M_\lambda$

$= \{0\}$ if $\underset{\deg a_{\{n\}}}{\underset{\shortparallel}{\underline{\Delta_a - n - 1}}} = -r < 0$.)

Hence M_{top} is a simple $A(V)$-module

(if $N \subset M_{top}$ $A(V)$-submodule, $U(V) \cdot N$ is a submodule of $M \overset{M \text{ simple}}{\Longrightarrow} N = M_{top}$

Conversely, let E be a simple A(V)-module

Since $M = \mathrm{Ind}_{A(V)}^{\mathcal{U}(V)} E$ is a positive energy repr. of V.

and $\left(\mathrm{Ind}_{A(V)}^{U(V)} E\right)_{top} = \mathcal{U}(V)_0 \otimes_{\mathcal{U}(V)_{\leq 0}} E \simeq E$

<u>Aim</u>: $\exists!$ graded quotient $L(E)$ of M.

Any graded proper submodule N of M intersects trivially E.

Indeed: if $N \cap E \neq \{0\}$, $m \in N \cap E = N_{top}$.

$E = A(V).m \subset N$ since E is a simple $A(V)$-module

But: E generates M as V-module because: $\mathrm{Ind}_{A(V)}^{U(V)} E = \mathcal{U}(V)_{>0}.E$.

$\Rightarrow N = M$.

Hence: $\exists!$ graded quotient $L(E)$ of M.

. Clearly the maps $E \longmapsto L(E)$ and $M \longmapsto M_{top}$ are inverse each others $\boxtimes$

Sketch of proof of thm 2 "$A(V) \simeq$ Zhu V"

[He: Xiao He: 2017, "Higher level Zhu algebras are subquotients of universal enveloping alg"

[Zhu, 1996 "Modular invariance of characters of VOA"]

Set $J_n(a) = a_{\{\Delta_a - 1 + n\}}$ deg $J_n(a) = \Delta_a - (\Delta_a - 1 + n) - 1 = \underline{\underline{-n}}$

Lemma: every element $a = J_{n_1}(a_1) \dots J_{n_m}(a_m)$ can be expressed in the quotient $A(V)$ as $J_0(v = v(a))$ where $v = v(a) \in V$

$$\begin{tikzcd}
V \arrow[r] \arrow[d] \arrow[rr, bend left, "\varphi"] & \mathcal{U}(V)_0 \text{"} = \overline{\mathcal{U}(\mathrm{Lie}\,V)_0} / I \text{"} \arrow[r] & A(V) = \mathcal{U}(V)_0 \Big/ \overline{\sum_{r>0} \mathcal{U}(V)_r\, \mathcal{U}(V)_{-r}} \\
v \arrow[r, mapsto] & \overline{v \otimes 1} \arrow[r, mapsto] & \text{image in the quotient} \\
\mathrm{Zhu}(V) = V/_{V \circ V} . & &
\end{tikzcd}$$

. By the technical lemma, φ is surjective.

. We have to show that φ factorizes through $\mathrm{Zhu}(V) = V/_{V \circ V}$

ie $\varphi(V \circ V) \subset \overline{\sum_{r>0} \mathcal{U}(V)_r\, \mathcal{U}(V)_{-r}}$

$$\text{"}\varphi\text{"}(a \circ b) = \sum_{i \geq 0} \binom{\Delta_a}{i} (a_{(i-1)} b)_{\{\Delta_a + \Delta_a - i\}} \overset{(B2)}{=} \sum_{i \geq 0} (-1)^i \binom{-2}{i} \left(a_{\{\Delta_a - 2 - i\}} b_{\{\Delta_b + i\}} - \dots\right)$$

$$= \sum_{i \geq 0} (-1)^i \binom{-2}{i} \left(\underset{\deg\, i+1}{J_{-i-1}}(a)\ \underset{-i-1}{J_{i+1}}(b) - J_{-i-1}(b)\ \underset{-i-1}{J_{i+1}}(a)\right)$$

$$\subset \overline{\sum_{r>0} \mathcal{U}(V)_r\, \mathcal{U}(V)_{-r}}$$

$\varphi : \mathrm{Zhu}(V) \longrightarrow A(V)$ well-defined.

. φ is an alg. homom. ie: $\varphi(a * b) = \varphi(a)\varphi(b)$.

$$\varphi(a*b) \underset{(B2)}{=} \dots = \sum_{i\geq 0} (-1)^i \binom{-1}{i} \Big(\underbrace{J_{-i}(a)\, J_i(b)}_{\deg\, -i} - J_{-i-1}(b)\, \underset{\deg\, -i-1\, <0}{J_{i+1}}(a) \Big)$$

$$= J_0(a)\, J_0(b) \mod \overline{\sum_{r>0} U(V)_r\, U(V)_{-r}}$$

On the other hand (proof of the technical lemma)

$J_0(a)\, J_0(b) = \dots = \varphi(a * b)$.

$*$ It remains to construct an inverse map.

By the technical lemma, any element of $A(V)$ is expressed as

$J_0(v) \mod \overline{\sum_{r>0} U(V)_r\, U(V)_{-r}}$

$\Upsilon : A(V) \longrightarrow \mathrm{Zhu}(V)$, $J_0(v) \mod .. \longmapsto v \mod V\circ V$

well-definedness requires that $J_0(v) \in \overline{\sum_{r>0} U(V)_r\, U(V)_{-r}}$, then $v \in V\circ V$.

(technical). $\boxtimes$

Forgot to prove:

Thm: If V admits a PBW basis, then $\eta_V : R_V \xrightarrow{\sim} \mathrm{gr}\,\mathrm{Zhu}\,V$ is an isom.

proof: $\mathrm{gr}\,\mathrm{Zhu}\,V = \mathrm{gr}\,V / \mathrm{gr}(V \circ V)$ where $\mathrm{gr}(V \circ V)$ is the graded space of $V \circ V$

w.r.t. to the filtration $(V_{\leq p})_p$

We wish to show that $\mathrm{gr}(V \circ V) = C_2(V)$

- Since $a \circ b = \underbrace{a_{(-2)} b}_{\text{deg } \Delta_a + \Delta_b + 1}$ mod $V_{\Delta_a + \Delta_b}$, it is sufficient to show that

$a \circ b \neq 0 \Rightarrow a_{(-2)} b \neq 0.$

Suppose $a_{(-2)} b = (Ta)_{(-1)} b = 0$ of b homog.

Since V admits a PBW basis, $\mathrm{gr}_F V$ has no zero divisor, we have $Ta = 0$.

Also from PBW property, $Ta = 0 \Rightarrow a = \lambda |0\rangle$, $\lambda \in \mathbb{C}$

↑ a lin comb of monomials $\underbrace{a^{i_1}_{(n_1)} \cdots a^{i_r}_{(n_r)}}_{\text{order on these monials.}} |0\rangle$

$\Rightarrow a \circ b = \lambda (|0\rangle \circ b) = \sum_{i \geq 0} \lambda \binom{0}{2} |0\rangle_{(i-2)} b = 0$ $\square$.

Rem: in general, $\eta_V : R_V \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}\,V$

In particular, if V is lisse, then R_V is finite-dimensional and so is $\mathrm{Zhu}(V)$.